MATH 20D Spring 2023 Lecture 16.

Properties of the Laplace Transform

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- Lecture for Friday May 12th and Monday May 15th will be recorded asynchronously and uploaded to Canvas. There will be **no** in person lecture on Friday May 12th and Monday May 15th.

Outline

Contents

Last Time

- $f\colon [0,\infty) \to \mathbb{R}$, the **Laplace Transform** $\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt$,

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Let $f\colon [0,\infty) \to \mathbb{R}$ be a function. If $0 \leqslant a \leqslant b < \infty$ then the definite integral

$$\int_a^b f(t)dt$$

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then $\mathscr{L}\{f\}(s) := \lim_{N\to\infty} \int_0^N e^{-st} f(t)dt$ converges for all $s > \alpha$.

Example

Determine which of the following functions satisfy the hypotheses of the theorem on the previous slide.

(a) $f(t) = \begin{cases} 1/(t-1), & t \neq 1 \\ 0, & t = 1 \end{cases},$ **(b)** $f(t) = \begin{cases} 1, & 0 \leqslant t < 5 \\ e^{t^2}, & 5 \leqslant t < \infty. \end{cases}$

(c) $f(t) = \begin{cases} e^{t^2}, & 0 \leqslant t < 30 \\ e^{3t}\sin(2t), & t \geqslant 30. \end{cases}$

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The Translation Property of Laplace Transform

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If the Laplace transform of a function f exists for $s > \alpha$, then

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Example

Let $a \in \mathbb{R}$ be constant, $\omega > 0$, and $n \in \mathbb{Z}_{\geqslant 1}$.

(a) Given that $\mathscr{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$ for $s > 0$, calculate $\mathscr{L}\{e^{at}\sin(\omega t)\}(s)$.

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(b) Given that $\mathscr{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$ for $s > 0$, calculate $\mathscr{L}\{e^{at}t^n\}(s)$.

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If $s > \alpha$ then

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First Derivatives in t-space

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Example

Let $\omega > 0$ be constant. Given that

$$\mathscr{L}\{\sin(\omega t)\}(s) = \frac{\omega}{s^2+\omega^2}, \qquad s > 0$$

Calculate

(a) $\mathscr{L}\{\cos(\omega t)\}$, **(c)** $\mathscr{L}\{\sin^2(\omega t)\}$, **(c)** $\mathscr{L}\{\cos^2(\omega t)\}$.

Recursively applying the formula $\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$ we obtain.

Corollary

Suppose $f\colon [0.\infty) \to \mathbb{R}$ is a function such that

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Corollary

Suppose $f\colon [0.\infty) \to \mathbb{R}$ is a function such that

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In particular $\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$.

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Example

Given that

$$\mathscr{L}\{t^{5/2}\} = \frac{15\sqrt{\pi}}{8s^{7/2}}, \qquad s > 0$$

Calculate the Laplace transform $\mathscr{L}\{\sqrt{t}\}$.